

Detecting Full N -Particle Entanglement in Arbitrarily High-Dimensional Systems with Bell-Type Inequality

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We derive a set of Bell-type inequalities for arbitrarily high-dimensional systems, based on the assumption of partial separability in the hybrid local-nonlocal hidden variable model. Partially entangled states would not violate the inequalities, and thus upon violation, these Bell-type inequalities are sufficient conditions to detect the full N -particle entanglement and invalidity of the hybrid local-nonlocal hidden variable description.

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I. INTRODUCTION

Entanglement is one of the most fundamental features of quantum mechanics, and it lies at the heart of recent quantum information theory. As a result, many remarkable achievements, such as quantum teleportation [1] and the higher levels of security in cryptography [2] have been attained owing to the quantum entanglement.

Given N -particle quantum systems, the correlations among them have been the subject of several recent studies [3–8]. This is also motivated by the question whether the correlations in recent experiments on 3- or 4-particle systems are due to the full N -particle entanglement and not just combinations of quantum entanglement of smaller number of particles [9, 10]. For $N = 2$, the entanglement type of the bipartite system is humdrum, i.e., it is either entangled or separable. However, the situation is dramatically changed when $N \geq 3$, besides the totally separable states, there are partially entangled states and fully N -particle entangled states. Consider all possible decompositions of a N -particle state as a mixture of pure states $\rho^N = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, if for any decomposition there is at least one $|\Psi_i\rangle$ showing N -particle entanglement, then we shall say that ρ^N exhibits full N -particle entanglement; if the state is not separable or not fully N -particle entangled, then we call that ρ^N is partially entangled.

Entanglement has been studied extensively in connection with the Bell inequality. The Bell inequality was originally proposed to ruled out local realism description of quantum mechanics [11]. This presents us a concept, the so-called non-locality, which is revealed by violations of the Bell inequality. Generally, entanglement and nonlocality are two different concepts. Although there exist entangled states not violating the known Bell inequalities, the violation of Bell inequality means the studied system is entangled, allowing us to detect entanglement. The conventional “ N -particle Bell inequality

ties” are designed to deny the local hidden variable (LHV) models [12–14]. For N -particle quantum systems, the partially entangled states and the fully N -particle entangled states may violate the same Bell inequality, namely, the conventional Bell inequalities do not distinguish at all the partially entangled states and the fully N -particle entangled states. Actually, a particle can decay into several particles, this gives rise to a natural question: Are the resultant decaying systems in a fully entangled state or just a partially entangled state? In 1987, Svetlichny triggered the problem and proposed a Bell-type inequality to distinguish full three-qubit entanglement from partially two-qubit ones [3]. The Svetlichny inequality is essentially different from the conventional Bell inequality, because the former is designed for a hybrid local-nonlocal hidden variable (HLNHV) model and the latter is for a LHV model. As the name implies, HLNHV models utilize the fusion of local and nonlocal descriptions based on the assumption of partial separability. Fifteen years later, Seevinck *et al.* and Collins *et al.* independently generalized the Svetlichny inequality from three-qubit case to arbitrarily N -qubit case [4, 5]. Upon violation, these N -qubit Bell-type inequalities are sufficient conditions for detecting full N -qubit entanglement.

In this paper, we generalize the Svetlichny inequalities from the qubit case to N arbitrarily d -dimensional systems (N -qudit). These Bell-type inequalities are derived based on the assumption of partial separability, or more generally speaking, on the so-called HLNHV models, thus the quantum mechanical violations of these inequalities provide experimentally accessible conditions to detect the full N -qudit entanglement and rule out the HLNHV models. The paper is organized as follows. We present the N -qudit Bell type inequality based on the HLNHV model in Sec. II. The proof of the N -qudit inequality begins with the cases of $N = 3, 4$, and the result is generalized to the case of arbitrary N . In Sec. III, we investigate quantum violation of the N -qudit inequality. We show that the Greenberger-Horne-Zeilinger (GHZ) states violate our Bell type inequality and find the explicit form of the violation which depends on particle number N and dimension d . We also investigate noise resistance of the inequality by using the so-called critical visibility. It is found that our Bell-

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type inequality is more noise resistant than the Svetlichny one for N qudits when $d \geq 3$. We end with conclusions in the last section.

II. N -QUDIT BELL-TYPE INEQUALITY

Consider an experimental situation involving N particles in which two measurements $i_n = 1, 2$ ($n = 1, \dots, N$) can be performed on each particle. Each of the measurements has d possible outcomes: $x_{i_n} = 0, 1, \dots, d-1$. We now follow Svetlichny's splendid ideas [3, 4] and make the following assumption of partial separability: The N -qudit system is composed of many subsystems, which might be correlated in any way (e.g. entangled) but are uncorrelated with respect to each other. Since we can always take any two subsystems jointly as a single one but still uncorrelated with respect to the others, we only need to focus on the case that the composed system consists of only two uncorrelated subsystems involving $m < N$ and $N - m < N$ qudits, respectively. For simplicity, we also assume that the first subsystem is formed by the first m qudits and the other by the remaining qudits. Denote the probability of observing the results x_{i_n} by $P(x_{i_1}, x_{i_2}, \dots, x_{i_N})$, then the partial separability assumption can be expressed as

$$P(x_{i_1}, x_{i_2}, \dots, x_{i_N}) = \int_{\Gamma} P_1(x_{i_1}, x_{i_2}, \dots, x_{i_m}|\lambda) \times P_2(x_{i_{m+1}}, x_{i_{m+2}}, \dots, x_{i_N}|\lambda) \rho(\lambda) d\lambda, \quad (1)$$

where $P_1(x_{i_1}, x_{i_2}, \dots, x_{i_m}|\lambda)$ and $P_2(x_{i_{m+1}}, \dots, x_{i_N}|\lambda)$ are probabilities conditioned to the hidden variable λ ; Γ is the total λ space and $\rho(\lambda)$ is a statistical distribution of λ , which satisfies $\rho(\lambda) \geq 0$ and $\int_{\Gamma} d\lambda \rho(\lambda) = 1$. Other decompositions can be described with a different value of m and different choices of the composing qudits. A HLNHV model can then be well defined based on the assumption of partial separability and the formula of the factorizable probability, readers who are interested in it may refer to Refs. [5, 15]. If the probability factorization can be

$$P(x_{i_1}, \dots, x_{i_N}) = \int_{\Gamma} d\lambda \rho(\lambda) P_1(x_{i_1}|\lambda) \cdots P_N(x_{i_N}|\lambda), \quad (2)$$

the HLNHV model then reduces to the usual LHV model.

For convenience, we introduce two functions:

$$\begin{aligned} g_1(x + s_t) &= \frac{S - M(x + s_t, d)}{S}, \\ g_2(x + s_t) &= \frac{S - M(-x - s_t, d)}{S}. \end{aligned} \quad (3)$$

Here $S = \frac{d-1}{2}$ is the spin value of the particle; s_t means a shift of the argument x ; $M(x, d) = (x \bmod d)$ and $0 \leq M(x, d) \leq d-1$. The N -qudit Bell-type inequality reads

$$I^N = - \left(\sum_{i_1, i_2, \dots, i_N=1}^2 Q_{i_1 i_2 \dots i_N} \right) \leq 2^{N-1}, \quad (4)$$

with

$$Q_{i_1 i_2 \dots i_N} \equiv \sum_{x_{i_1}, \dots, x_{i_N}=0}^{d-1} f^{i_1 i_2 \dots i_N} P(x_{i_1}, x_{i_2}, \dots, x_{i_N}). \quad (5)$$

Let $\mathcal{I} \equiv i_1 i_2 \dots i_N$, and $t(\mathcal{I})$ denotes the times that the index "2" appears in the string \mathcal{I} , we then abbreviate the coefficient as follows:

$$\begin{aligned} f^{\mathcal{I}}(x_{i_1}, x_{i_2}, \dots, x_{i_N}, s_t) &\equiv g_1 \\ &= 1 - M(x_{i_1} + \dots + x_{i_N} + s_t, d)/S \end{aligned} \quad (6)$$

if $t(\mathcal{I})$ is even and

$$\begin{aligned} f^{\mathcal{I}}(x_{i_1}, x_{i_2}, \dots, x_{i_N}, s_t) &\equiv g_2 \\ &= 1 - M(-x_{i_1} - \dots - x_{i_N} - s_t, d)/S \end{aligned} \quad (7)$$

if $t(\mathcal{I})$ is odd, and $s_t \equiv s_{t(\mathcal{I})} = 3 \times (1 - [\frac{t(\mathcal{I})}{2}])$, where $[\frac{t}{2}]$ means the integer part of $\frac{t}{2}$. The above inequality is symmetric under permutations of the N particles. Essentially, the inequality (4) is a kind of probabilistic Bell-type inequality if substitute Eq. (5) into inequality (4). We express it in the form of inequality (4) for two reasons: (i) to make the inequality succinct and (ii) $Q_{\mathcal{I}}$ may be regarded as generalized correlation functions of N -qudit in comparison to the typical form of correlation functions of qubits. As for the coefficient $f^{\mathcal{I}}$, its possible values are equal to $S_z/S \in \{-1, -1+1/S, \dots, 1\}$, where S_z is expectation value of the z -component of the spin operators. Especially $f^{\mathcal{I}}$ has only two possible values ± 1 when $S = 1/2$, $Q_{\mathcal{I}}$ reduces to the typical form of correlation functions of qubits.

In the following, we shall prove that the upper bound of the inequality is 2^{N-1} .

Three qudits.—Our inequality for three qudits reads

$$\begin{aligned} I^3 &= -Q_{111} - Q_{112} - Q_{121} - Q_{122} \\ &\quad - Q_{211} - Q_{212} - Q_{221} - Q_{222} \leq 4, \end{aligned} \quad (8)$$

with $f^{111} = f^{122} = f^{221} = f^{212} = g_1$ and $f^{222} = f^{112} = f^{121} = f^{211} = g_2$; the shift $s_{t(111)} = s_{t(112)} = s_{t(121)} = s_{t(211)} = 3$ and $s_{t(122)} = s_{t(212)} = s_{t(221)} = s_{t(222)} = 0$. We assume that, for three qudits, the two uncorrelated subsystems are the first two qudits and the third qudit. Hence, in a HLNHV model with two-setting scenario, four possible outcomes for the first two qudits $x_{i_1} + x_{i_2}$ and two outcomes for the third qudit x_{i_3} are independent of each other. We then simply denote $x_{i_1} + x_{i_2}$ by a single variable ξ_{ij} and x_{i_3} by ζ_k , both are from 0 to $d-1$. Moreover, since any nondeterministic local variable model can be made deterministic by adding additional variables [16], we only need to consider the *deterministic* versions [17] of the HLNHV model in which for each value of λ , the measurement outcomes are completely determined, namely, the probability of obtaining each possible outcome is either 0 or 1. For each λ , we have predetermined values for the outcomes of ξ_{ij} and ζ_k ($i, j, k = 1, 2$).

Another preliminary knowledge is about our two-qudit inequality

$$I^2 = -Q_{11} - Q_{12} - Q_{21} - Q_{22} \leq 2, \quad (9)$$

where $f^{11} = f^{22} = g_1$, $f^{12} = f^{21} = g_2$, $s_{t(11)} = s_{t(12)} = s_{t(21)} = 3$, and $s_{t(22)} = 0$. From the definition of $Q_{i_1 i_2}$, we have the explicit form of inequality (9) as

$$I^2 = -g_1(r_{11} + 3) - g_2(r_{12} + 3) - g_2(r_{21} + 3) - g_1(r_{22}) \quad (10)$$

here $r_{ij} \equiv \alpha_i + \beta_j$, α_i being the outcome of the first qudit for the i -th measurement and β_j being that of the second qudit for the j -th measurement. In the following, we show that inequality (9) is an equivalent form of the well-known Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [18], which is usually of the form

$$I_{\text{CGLMP}} = Q_{11} + Q_{12} + Q_{21} - Q_{22} \leq 2. \quad (11)$$

In our notation, the CGLMP inequality can be rewritten as

$$I_{\text{CGLMP}} = g_2(r'_{11}) + g_1(r'_{12}) + g_1(r'_{21}) - g_1(r'_{22}), \quad (12)$$

with $r'_{ij} \equiv \alpha'_i + \beta'_j$. By using $g_1(x) = -g_2(x + 1)$, inequality (9) [or (10)] is of the form

$$I^2 = g_2(r_{11} + 4) + g_1(r_{12} + 2) + g_1(r_{21} + 2) - g_1(r_{22}) \quad (13)$$

Let $\alpha'_1 = \alpha_1 + 2$, $\alpha'_2 = \alpha_2$, $\beta'_1 = \beta_1 + 2$, and $\beta'_2 = \beta_2$, one immediately finds that I^2 and I_{CGLMP} are of the same form. Hereafter we simply call inequality (9) as the CGLMP inequality. The proof of inequality (4) resorts to $I^2 \leq 2$.

To prove $I^3 \leq 4$, we write $I^3 = \mathbb{I}_1 + \mathbb{I}_2$ with

$$\begin{aligned} \mathbb{I}_1 &\equiv -Q_{111} - Q_{112} - Q_{121} - Q_{122} \\ \mathbb{I}_2 &\equiv -Q_{211} - Q_{212} - Q_{221} - Q_{222}. \end{aligned} \quad (14)$$

According to the definition of $Q_{i_1 i_2 i_3}$, we have

$$\begin{aligned} \mathbb{I}_1 &= -g_1(\xi_{11} + \zeta_1 + 3) - g_2(\xi_{11} + \zeta_2 + 3) \\ &\quad - g_2(\xi_{12} + \zeta_1 + 3) - g_1(\xi_{12} + \zeta_2). \end{aligned} \quad (15)$$

From Eq. (10), if one sets $\xi_{11} = \alpha_1$, $\xi_{12} = \alpha_2$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, then one easily finds that \mathbb{I}_1 is equivalent to I^2 , thus $\mathbb{I}_1 \leq 2$. Similarly, we find

$$\begin{aligned} \mathbb{I}_2 &= -g_2(\xi_{21} + \zeta_1 + 3) - g_1(\xi_{21} + \zeta_2) \\ &\quad - g_1(\xi_{22} + \zeta_1) - g_2(\xi_{22} + \zeta_2), \end{aligned} \quad (16)$$

and set $\xi_{22} = \alpha_1 + 3$, $\xi_{21} = \alpha_2$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, then \mathbb{I}_2 is equivalent to I^2 , so $\mathbb{I}_2 \leq 2$. Thus we have $I^3 = \mathbb{I}_1 + \mathbb{I}_2 \leq 4$.

Four qudits.—Our inequality for four qudits reads

$$\begin{aligned} I^4 &= -Q_{1111} - Q_{1112} - Q_{1121} - Q_{1211} - Q_{2111} \\ &\quad - Q_{1122} - Q_{2211} - Q_{1212} - Q_{2121} - Q_{1221} \\ &\quad - Q_{2112} - Q_{1222} - Q_{2122} - Q_{2212} - Q_{2221} \\ &\quad - Q_{2222} \leq 8, \end{aligned} \quad (17)$$

with $f^{1111} = f^{1122} = f^{2211} = f^{1212} = f^{2121} = f^{1221} = f^{2112} = f^{2222} = g_1$, and the others are g_2 ; the shift $s_{t(1111)} = s_{t(1112)} = s_{t(1121)} = s_{t(1211)} = s_{t(2111)} = 3$, $s_{t(2222)} = -3$, and the others are zero. The process to obtain the upper bound

is similar to that of the three qudits. We first write the 4-qudit inequality as $I^4 = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4$ with

$$\begin{aligned} \mathbb{I}_1 &\equiv -Q_{1111} - Q_{1112} - Q_{1211} - Q_{1212}, \\ \mathbb{I}_2 &\equiv -Q_{2211} - Q_{2212} - Q_{2111} - Q_{2112}, \\ \mathbb{I}_3 &\equiv -Q_{1221} - Q_{1222} - Q_{1121} - Q_{1122}, \\ \mathbb{I}_4 &\equiv -Q_{2121} - Q_{2122} - Q_{2221} - Q_{2222}. \end{aligned} \quad (18)$$

For four qudits, the system may consist of three-qudit and one-qudit subsystems, or of two two-qudit subsystems when we study partial entanglement. For the former case, define $r_{ijkl} \equiv \xi_{ijk} + \zeta_l$ and write \mathbb{I}_1 as

$$\begin{aligned} \mathbb{I}_1 &= -g_1(r_{1111} + 3) - g_2(r_{1112} + 3) \\ &\quad - g_2(r_{1211} + 3) - g_1(r_{1212}) \\ &= -g_1(\xi_{111} + \zeta_1 + 3) - g_2(\xi_{111} + \zeta_2 + 3) \\ &\quad - g_2(\xi_{121} + \zeta_1 + 3) - g_1(\xi_{121} + \zeta_2). \end{aligned} \quad (19)$$

If we set $\xi_{111} = \alpha_1$, $\xi_{121} = \alpha_2$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, we find that \mathbb{I}_1 is equivalent to I^2 and so $\mathbb{I}_1 \leq 2$; similarly if we set $\xi_{221} = \alpha_1 + 3$, $\xi_{211} = \alpha_2$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, then $\mathbb{I}_2 \leq 2$; if we set $\xi_{122} = \alpha_1 + 3$, $\xi_{112} = \alpha_2$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, then $\mathbb{I}_3 \leq 2$; if we set $\xi_{212} = \alpha_1 + 3$, $\xi_{222} = \alpha_2 + 3$, $\zeta_1 = \beta_1$, $\zeta_2 = \beta_2$, then $\mathbb{I}_4 \leq 2$, and thus we find $I^{[4]} = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 \leq 8$. For the latter, we define $r_{ijkl} \equiv \xi_{ij} + \zeta_{kl}$. If we set $\xi_{11} = \alpha_1$, $\xi_{12} = \alpha_2$, $\zeta_{11} = \beta_1$, $\zeta_{12} = \beta_2$, then $\mathbb{I}_1 \leq 2$; if we set $\xi_{22} = \alpha_1 + 3$, $\xi_{21} = \alpha_2$, $\zeta_{11} = \beta_1$, $\zeta_{12} = \beta_2$, then $\mathbb{I}_2 \leq 2$; if we set $\xi_{12} = \alpha_1 + 3$, $\xi_{11} = \alpha_2$, $\zeta_{21} = \beta_1$, $\zeta_{22} = \beta_2$, then $\mathbb{I}_3 \leq 2$; if we set $\xi_{21} = \alpha_1 + 3$, $\xi_{22} = \alpha_2 + 3$, $\zeta_{21} = \beta_1$, $\zeta_{22} = \beta_2$, then $\mathbb{I}_4 \leq 2$. Thus $I^{[4]} = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4 \leq 8$.

Arbitrary N qudits.—Based on the CGLMP inequality (9), now we can prove the N -qudit Bell-type inequality (4) as what we have done in the three- and four-qudit cases. For further convenience and without losing the generalization, we abbreviate the correlation function $-Q_{\mathcal{I}}$ by (k) , here $k = t(\mathcal{I})$. For examples, (0) means that $i_1 = \dots = i_N = 1$ in $Q_{i_1 i_2 \dots i_N}$, or $(0) = -Q_{11 \dots 1}$; (1) means that one of the index $\{i_1, i_2, \dots, i_N\}$ is 2 and the others are 1, or $(1) = -Q_{21 \dots 1} / -Q_{12 \dots 1} / \dots / -Q_{11 \dots 2}$; and (k) represents the correlation function $-Q_{\mathcal{I}}$ in which there are k “2” and $(N - k)$ “1” in the index \mathcal{I} . It is easy to see that the number of (0) is C_N^0 , that of (1) is C_N^1 , and so on. In this language, the four-qudit Bell-type inequality can be expressed as $I^4 = \sum_{j=1}^4 \mathbb{I}_j \leq 8$ with

$$\begin{aligned} \mathbb{I}_1 &\equiv (0) + (1) + (1) + (2), \\ \mathbb{I}_2 &\equiv (1) + (2) + (2) + (3), \\ \mathbb{I}_3 &\equiv (1) + (2) + (2) + (3), \\ \mathbb{I}_4 &\equiv (2) + (3) + (3) + (4). \end{aligned} \quad (20)$$

For $N = 4$, the 2^4 correlation functions are divided into four subgroups with each subgroup possessing the feature of

$$\mathcal{G}(k) \equiv (k) + (k + 1) + (k + 1) + (k + 2) \quad (21)$$

as shown above. One more thing worth to note is that \mathbb{I}_j are grouped according to the index string $i_1 i_2 i_3 i_4$. Take \mathbb{I}_1 as an

example, $\mathbb{I}_1 = -(Q_{111} + Q_{112}) \otimes (Q_1 + Q_2)$ for the case that the four-qudit system consists of three-qudit and one-qudit subsystems; and $\mathbb{I}_1 = -(Q_{11} + Q_{12}) \otimes (Q_{11} + Q_{12})$ for the case that the system consists of two two-qudit subsystems. One can find $t(111) + 1 = t(112)$ and $t(1) + 1 = t(2)$ for the former case, and $t(11) + 1 = t(12)$ for the latter case. Similar results can be obtained for other \mathbb{I}_j .

Now let us look at N qudits. First we show that the N -qudit Bell-type inequality can be rearranged into a grouping with each subgroup of the form $\mathcal{G}(k)$, that is

$$I^N = \sum_{k=0}^{N-2} T(k) \times \mathcal{G}(k), \quad (22)$$

here $T(k)$ indicates the times that the element $\mathcal{G}(k)$ appears. We obtain iterative equations for the coefficients $T(k)$

$$\begin{aligned} T(0) &= C_N^0 = 1, \\ T(1) &= C_N^1 - 2T(0) = C_N^1 - 2C_N^0, \\ T(2) &= C_N^2 - 2T(1) - T(0) = C_N^2 - 2C_N^1 + 3C_N^0, \\ &\vdots \\ T(k) &= \sum_{i=0}^k (-1)^{k-i} (k+1-i) C_N^i. \end{aligned} \quad (23)$$

The summation of $T(k)$ yields $\sum_{k=0}^{N-2} T(k) = 2^{N-2}$. Since there are four terms in each $T(k)$, so the total number of terms $4 \sum_{k=0}^{N-2} T(k) = 2^N$ is exactly the number of terms in a N -particle inequality for two settings. Therefore such a rearrangement always exists in our inequality.

Secondly, consider a N -qudit system consisting of two subsystems of m qudits and $N - m$ qudits. According to our two-setting scenario, in subsystem m we have a set of 2^m index strings $i_1 i_2 \dots i_m$ and in subsystem $N - m$ we have a set of 2^{N-m} index strings $i_{m+1} \dots i_N$; by connection of index strings of two subsystems we have totally $2^m \cdot 2^{N-m} = 2^N$ index strings $i_1 i_2 \dots i_m i_{m+1} \dots i_N$ indicating the kind of measurement for each qudit in the whole system. This implies the correlation function $Q_{i_1 \dots i_m i_{m+1} \dots i_N} = Q_{i_1 \dots i_m} \otimes Q_{i_{m+1} \dots i_N}$. Let the four correlation functions in each subgroup be of the form

$$\begin{aligned} &-Q_{i_1 \dots i_m i_{m+1} \dots i_N} - Q_{i_1 \dots i_m i'_{m+1} \dots i'_N} \\ &-Q_{i'_1 \dots i'_m i_{m+1} \dots i_N} - Q_{i'_1 \dots i'_m i'_{m+1} \dots i'_N}. \end{aligned} \quad (24)$$

As stated above, the rule for grouping correlation functions is based on the index strings. The general rule is

$$\begin{aligned} t(i_1 i_2 \dots i_m) + 1 &= t(i'_1 i'_2 \dots i'_m), \\ t(i_{m+1} \dots i_N) + 1 &= t(i'_{m+1} \dots i'_N). \end{aligned} \quad (25)$$

Use the property $t(i_1 \dots i_m i_{m+1} \dots i_N) = t(i_1 \dots i_m) + t(i_{m+1} \dots i_N)$, and let $k = t(i_1 \dots i_m i_{m+1} \dots i_N)$, we find $k + 1 = t(i_1 \dots i_m i'_{m+1} \dots i'_N) = t(i'_1 \dots i'_m i_{m+1} \dots i_N)$, and $k + 2 = t(i'_1 \dots i'_m i'_{m+1} \dots i'_N)$, we then exactly have the simple form of $\mathcal{G}(k)$ from the expression (24).

Next we show that $\mathcal{G}(k) \leq 2$ by considering two cases of k = even and k = odd.

Case a: k is even, we obtain

$$\begin{aligned} \mathcal{G}(k) &= \\ &g_1(r_{i_1 \dots i_m i_{m+1} \dots i_N} + 3 - 3[k/2]) \\ &+ g_2(r_{i_1 \dots i_m i'_{m+1} \dots i'_N} + 3 - 3[(k+1)/2]) \\ &+ g_2(r_{i'_1 \dots i'_m i_{m+1} \dots i_N} + 3 - 3[(k+1)/2]) \\ &+ g_1(r_{i'_1 \dots i'_m i'_{m+1} \dots i'_N} + 3 - 3[(k+2)/2]). \end{aligned} \quad (26)$$

As the N -qudit system consists of two subsystems of m qudits and $N - m$ qudits, we have $r_{i_1 i_2 \dots i_N} = \xi_{i_1 \dots i_m} + \zeta_{i_{m+1} \dots i_N}$. If we set $\xi_{i_1 \dots i_m} = \alpha_1 + 3a$, $\xi'_{i'_1 \dots i'_m} = \alpha_2 + 3a$, $\zeta_{i_{m+1} \dots i_N} = \beta_1$, $\zeta'_{i'_{m+1} \dots i'_N} = \beta_2$, and $a = k/2$, then this group of correlation functions is equivalent to the CGLMP inequality (9) and thus its upper bound is 2.

Case b: k is odd, we have

$$\begin{aligned} \mathcal{G}(k) &= \\ &g_1(\xi'_{i'_1 \dots i'_m} + \zeta_{i_{m+1} \dots i_N} + 3 - 3[(k+1)/2]) \\ &+ g_2(\xi'_{i'_1 \dots i'_m} + \zeta'_{i'_{m+1} \dots i'_N} + 3 - 3[(k+2)/2]) \\ &+ g_2(\xi_{i_1 \dots i_m} + \zeta_{i_{m+1} \dots i_N} + 3 - 3[k/2]) \\ &+ g_1(\xi_{i_1 \dots i_m} + \zeta'_{i'_{m+1} \dots i'_N} + 3 - 3[(k+1)/2]). \end{aligned} \quad (27)$$

If we set $\xi_{i_1 \dots i_m} = \alpha_2 + 3(b-1)$, $\xi'_{i'_1 \dots i'_m} = \alpha_1 + 3b$, $\zeta_{i_{m+1} \dots i_N} = \beta_1$, $\zeta'_{i'_{m+1} \dots i'_N} = \beta_2$, and $b = (k+1)/2$, then $\mathcal{G}(k)$ is equivalent to the CGLMP inequality (9) and thus its upper bound is 2.

Based on the above analysis, such a group of correlation functions $\mathcal{G}(k)$ is always less than 2. There are totally 2^{N-2} subgroups in I^N , therefore $I^N \leq 2^{N-1}$. This ends the proof.

III. QUANTUM VIOLATION OF THE N -QUDIT BELL-TYPE INEQUALITY

We now turn to study quantum violations of the inequality (4) for the GHZ states

$$|\psi\rangle_{\text{GHZ}}^N = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj \dots j\rangle, \quad (28)$$

which are fully entangled states of N -qudit. Quantum mechanical joint probability is calculated by

$$\begin{aligned} P^{\text{QM}}(x_{i_1}, x_{i_2}, \dots, x_{i_N}) &= \text{Tr}[\rho^N (U_{i_1} \otimes U_{i_2} \otimes \dots \otimes U_{i_N}) \\ &\times (\Pi_{i_1} \otimes \Pi_{i_2} \otimes \dots \otimes \Pi_{i_N}) (U_{i_1}^\dagger \otimes U_{i_2}^\dagger \otimes \dots \otimes U_{i_N}^\dagger)], \end{aligned} \quad (29)$$

where $\Pi_{i_n} = |x_{i_n}\rangle\langle x_{i_n}|$ and U_{i_n} ($n = 1, 2, \dots, N$) are projectors and unitary transformation operators for corresponding qudits. As for U_{i_n} , it is sufficient to consider the unbiased symmetric multi-port beamsplitters [19] when we study the GHZ states. A photon entering at any input ports of a symmetric d -port beam splitter has an equal chance as $1/d$ of exiting at any output ports. The action of the multipoint beam splitter can be described by a unitary transformation \mathcal{T} with elements

$\mathcal{T}_{kl} = \frac{1}{\sqrt{d}}\omega^{kl}$, where $\omega = \exp(\frac{i2\pi}{d})$. In front of the i -th input port, there is a phase shifter to adjust the phase of the incoming photon by ϕ_i . The phase shifts can be denoted as a d -dimensional vector $\vec{\phi} = (\phi_0, \phi_1, \dots, \phi_{d-1})$. The symmetric d -port beam splitter together with the d phase shifters perform the unitary transformation $U_{i_n}(\vec{\phi})$ with elements $U_{kl}(\vec{\phi}) = \frac{\omega^{kl} \exp(i\phi_l)}{\sqrt{d}}$. Specifically, take the first qudit as an example, the phase angles are $\vec{\phi}_{i_1=1} = (\phi_{10}, \phi_{11}, \dots, \phi_{1(d-1)})$ and $\vec{\phi}_{i_1=2} = (\phi_{20}, \phi_{21}, \dots, \phi_{2(d-1)})$ due to the two-setting scenario.

Numerical calculations show that maximal violation of the Bell-type inequality for the GHZ states can be found with the following optimal angle settings

$$\begin{aligned} \vec{\phi}_{i_n=1} &= \vec{\phi}_{i_1=1} \\ &= (0, \frac{m_1\pi}{2d}, 2\frac{m_1\pi}{2d}, \dots, (d-1)\frac{m_1\pi}{2d}), \end{aligned} \quad (30)$$

$$\begin{aligned} \vec{\phi}_{i_n=2} &= \vec{\phi}_{i_1=2} \\ &= (0, \frac{m_2\pi}{2d}, 2\frac{m_2\pi}{2d}, \dots, (d-1)\frac{m_2\pi}{2d}), \end{aligned} \quad (31)$$

where $m_1 = 15/N$, $m_2 = m_1 - 6$. For $N = 2$, our result correctly recovers that of the CGLMP inequality [18], namely

$$[I^2]^{\max} = 4d \sum_{k=0}^{[d/2]-1} (1 - \frac{2k}{d-1})(q_k - q_{-(k+1)}), \quad (32)$$

where $q_c = \frac{1}{2d^3 \sin^2[\pi(c+1/4)/d]}$. The maximal violations increase with dimension d , for examples, $[I_{d=2}^2]^{\max} = 2\sqrt{2} \simeq 2.828$ and $[I_{d=3}^2]^{\max} = (12 + 8\sqrt{3})/9 \simeq 2.873$. For arbitrary N -qudit, the maximal violation is

$$[I^N]^{\max} = 2^{N-2} \times [I^2]^{\max}. \quad (33)$$

One can show how sensitive the inequality (4) is by considering the factor \mathcal{R} defined by maximal violation of the inequality over upper bound for true N -body entanglement [8], i.e.,

$$\mathcal{R} = [I^N]^{\max} / 2^{N-1}. \quad (34)$$

It is easy to have $\mathcal{R} = \frac{[I^2]^{\max}}{2}$ for two qudits, and this confirms that inequality (4) is an equivalent form of the CGLMP inequality when $N = 2$. For $d = 2$, the Bell-type inequality (4) is an equivalent version of the Svetlichny inequality for N qubits, and accordingly $\mathcal{R} = \sqrt{2}$. Our result is in accordance with Refs. [4, 5] (see Eq. (14) in [5]).

So far we have discussed the violations of the pure GHZ states. If white noise is added, the pure state turns to a mixed state as

$$\rho^N(V) = V\rho_{\text{GHZ}} + (1-V)\rho_{\text{noise}}, \quad (35)$$

where $\rho_{\text{noise}} = \frac{1}{d^N} \mathbf{1}$ is the unit operator, V is the so-called visibility, and $0 \leq V \leq 1$. We find that the mixed state violates inequality (4) if $V > V_{\text{cr}}$, where $V_{\text{cr}} = \frac{1}{\mathcal{R}}$ is the critical

value of visibility. The critical values decrease when dimension d goes up, for examples, $V_{\text{cr}} = 0.707$ for $d = 2$ and $V_{\text{cr}} = 0.696$ for $d = 3$. The HLNHV description of the state is not allowed if $V > V_{\text{cr}}$. The less the value of V_{cr} is, the more noise tolerant the Bell inequality is. Moreover, for any partially entangled states

$$\rho^N = \rho^m \otimes \rho^{N-m}, \quad (36)$$

the quantum joint probability are factorizable, i.e.,

$$\begin{aligned} P^{\text{QM}}(x_{i_1}, x_{i_2}, \dots, x_{i_N}) &= P_1^{\text{QM}}(x_{i_1}, x_{i_2}, \dots, x_{i_m}) \\ &\times P_2^{\text{QM}}(x_{i_{m+1}}, x_{i_{m+2}}, \dots, x_{i_N}). \end{aligned} \quad (37)$$

Consequently, our inequality holds for any partially entangled states or any convex mixture of them. Any violation of our inequality is a sufficient condition to confirm full N -particle entanglement and rule out the HLNHV description. Let us point out that the N -qubit Svetlichny inequality can also be used to test the invalidity of the HLNHV description for N qudits by dividing the d outcomes into two sets, provided that $V > V_{\text{cr}}^s$ with V_{cr}^s being the critical visibility for the Svetlichny inequality. It is easy to see that $V_{\text{cr}}^s = \frac{1}{\sqrt{2}}$ does not depend on dimension d , and $V_{\text{cr}} < V_{\text{cr}}^s$ when $d \geq 3$. Thus our Bell-type inequality is more noise resistant than the Svetlichny one for N qudits when $d \geq 3$.

IV. CONCLUSION

Based on the assumption of partial separability we have derived a set of Bell-type inequalities for arbitrarily high-dimensional systems. Partially entangled states would not violate the inequalities, thus upon violation, the Bell-type inequalities are sufficient conditions to detect the full N -qudit entanglement and rule out the HLNHV description. It is observed that the Bell-type inequalities for multi-qudit ($d \geq 3$) violate the hybrid local-nonlocal realism more strongly than the Svetlichny ones for qubits, and the quantum violations increase with dimension d . Furthermore, how to generalize the Bell-type inequality to the multi-setting one remains a significant topic to be investigated further.

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Note added.— Ref. [20] also presents a similar inequality to our inequality (4).

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